PERMANENT CONFIGURATIONS IN THE n-BODY PROBLEM

BY

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1. **Introduction.** The problem of two bodies for spheres, homogeneous in concentric layers and finite in size, was first solved in a geometrical way by Newton [14](1) about 1685. Euler [4] gave the first detailed analytical solution of the problem in 1744. In 1772 Lagrange [8] gave four particular solutions of the three-body problem in his prize memoir. All solutions of the two-body problem and Lagrange's particular solutions of the three-body problem belong to that special class called permanent configurations(2), which we we shall define presently for the case of n-bodies.

Consider n free particles in space which attract each other along lines joining them according to any function of the distance. If all the masses are projected with initial velocities, and thereafter are acted upon only by the force of attraction, their orbits will be space curves. The problem of determining the positions and velocities of the bodies at any later time is the problem of n bodies. *A configuration of n bodies is said to be permanent if, as the masses move in their respective orbits, the ratios of the mutual distances remain constant.* Such a configuration may change in size but not in shape.

The literature on permanent configuration problems contains two methods of approach, (1) that of establishing continuity between the configurations for $n-1$ and n-bodies, and (2) that of characterizing the configurations for any n by the necessary and sufficient conditions which the $n(n-1)/2$ mutual distances must satisfy. Collinear configurations were discovered for the case $n=3$ by Euler [5] and for any n by Lehmann-Filhès [9] and F. R. Moulton [13], the latter using the first approach. Noncollinear configurations for the case $n=3$ were discovered by Lagrange [8], who also treated the collinear case. Dziobek [3] discussed the general approach (2) and arrived at some results for the case $n=4$, which case was treated in detail by MacMillan and Bartky [12]. W. L. Williams [16], applying the same method, considered the noncollinear case $n=5$. Some noncollinear solutions of the n-body problem have been found by Hoppe [7], Andoyer [1], Longley [10] and Emilia Breglia [2], but all have some element of symmetry.

Space permanent configurations are the scarcest and have the undesirable property that the motion of all bodies is either toward their common center

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) MacMillan [11, pp. 71–72, 75–78].

of gravity, in which case the configuration lasts for a finite time, or in the the opposite direction, in which case the configuration is expanding without limit(3). In fact it can be shown that all permanent configurations are either space configurations of the dilating without rotation type or they are planar(4). We shall treat only the plane configurations, and in particular only those rotating without dilation, for if the law of attraction is the Newtonian law and there exists a permanent configuration in which the n bodies revolve in concentric circles about the common center of gravity, the bodies may also move in similar conics with the common center of gravity as foci(5).

In most cases mentioned above it has not been necessary to restrict the masses or mass ratios, but in this paper we shall, upon applying approach (1), make certain restrictions which are sufficient, though they may not be necessary, to insure continuity.

2. **The equations for a permanent configuration.** The problem of n bodies belongs in the field of differential equations. If we require that the ratios of the mutual distances remain constant, the problem becomes one of permanent configurations and belongs in the theory of implicit functions. As pointed out in §1, we need require only that the motion be circular, that is, the distances themselves remain constant.

Let the origin be taken at the center of mass of the system, and let ω denote the constant angular velocity of the n masses about the origin. The equations for a permanent configuration are(6)

$$
\begin{aligned}
&\sum_{j=1}^{n}{}' \frac{x_i - x_j}{(r_{ij})^3} m_j - \lambda x_i = 0, \\
&\sum_{j=1}^{n}{}' \frac{y_i - y_j}{(r_{ij})^3} m_j - \lambda y_i = 0,
\end{aligned}
\qquad i = 1, 2, \cdots, n, \tag{2.1}
$$

where $\lambda = \omega^2$, $(r_{ij})^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ and the prime on the summation denotes the sum for all j except j equal i. It is a problem now in implicit function theory to solve equations (2.1) for the $2n$ coordinates x_i and y_i in terms of the $n+1$ parameters m_i and λ.

3. **Existence of a point of libration.** The first step in establishing continuity between the solutions for $n-1$ and n bodies consists of proving the existence of at least one point of libration for the case of $n-1$ bodies, and this section is devoted to that purpose.

Suppose that for a set of positive values $m_1, m_2, \cdots, m_{n-1}, \lambda$, and for $n \geqq 4$, the first $n-1$ equations of each set (2.1) have a solution (x_i^0, y_i^0)

(3) The space configuration for $n=4$ was first noted by Lehmann-Filhès [9]. Cf. also MacMillan [11, p. 74], and Wintner [17, p. 279].

(4) Wintner [17, pp. 287 ff.].

(5) Wintner [17, p. 300], or MacMillan [11, p. 74].

(6) Wintner [17, p. 302].

$(i=1, 2, \cdots, n-1)$ and that these points $P_i^0=(x_i^0, y_i^0)$ are not collinear. If an infinitesimal mass m_n be added to the system at the point (x_n, y_n) then x_n and y_n must satisfy the equations

$$\begin{aligned} \phi_1(x_n, y_n) &= \sum_{j=1}^{n-1} \frac{(x_n - x_j^0)m_j}{[(x_n - x_j^0)^2 + (y_n - y_j^0)^2]^{3/2}} - \lambda x_n = 0, \\ \phi_2(x_n, y_n) &= \sum_{j=1}^{n-1} \frac{(y_n - y_j^0)m_j}{[(x_n - x_j^0)^2 + (y_n - y_j^0)^2]^{3/2}} - \lambda y_n = 0. \end{aligned} \tag{3.1}$$

The subscript n may be omitted, and the two equations

$$\phi_1(x, y) = 0, \qquad \phi_2(x, y) = 0, \tag{3.2}$$

are the equations of two algebraic plane curves which we shall call C_1 and C_2 respectively. Since the $n-1$ points P_i^0 satisfy both equations (3.1), both curves pass through each of the P_i^0. It will be shown that C_1 and C_2 have at least one real intersection other than the $n-1$ points P_i^0.

In order to determine the behavior of the curves in the neighborhood of any one of the points P_i^0, we may write the equations for C_1 and C_2 in the form

$$\begin{aligned} G_1(x, y) &= (x - x_i^0)m_i + \left[(x - x_i^0)^2 + (y - y_i^0)^2\right]^{3/2} f_1(x, y) = 0, \\ G_2(x, y) &= (y - y_i^0)m_i + \left[(x - x_i^0)^2 + (y - y_i^0)^2\right]^{3/2} f_2(x, y) = 0, \end{aligned}$$

where

$$\begin{aligned} f_1(x, y) &= \sum_{j=1}^{n-1} \frac{(x - x_j^0)m_j}{[(x - x_j^0)^2 + (y - y_j^0)^2]^{3/2}} - \lambda x, \\ f_2(x, y) &= \sum_{j=1}^{n-1} \frac{(y - y_j^0)m_j}{[(x - x_j^0)^2 + (y - y_j^0)^2]^{3/2}} - \lambda y. \end{aligned} \tag{3.3}$$

Since $f_1(x_i^0, y_i^0)=0$, and $f_2(x_i^0, y_i^0)=0$ from the ith equation in each set (2.1), it follows that $f_1(x, y)$ and $f_2(x, y)$ may be expanded in powers of $x-x_i^0$ and $y-y_i^0$, vanishing for $x=x_i^0$, $y=y_i^0$. Thus

$$\begin{aligned} \phi_1(x, y) &= \frac{(x - x_i^0)m_i}{[(x - x_i^0)^2 + (y - y_i^0)^2]^{3/2}} \\ &\quad + A_1(x - x_i^0) + B_1(y - y_i^0) + \cdots, \\ \phi_2(x, y) &= \frac{(y - y_i^0)m_i}{[(x - x_i^0)^2 + (y - y_i^0)^2]^{3/2}} \\ &\quad + A_2(x - x_i^0) + B_2(y - y_i^0) + \cdots. \end{aligned} \tag{3.4}$$

The slope of C_1 is

$$-\frac{\partial G_1/\partial x}{\partial G_1/\partial y}=\frac{m_i x+3[(x-x_i^0)^2+(y-y_i^0)^2]^{1/2}(x-x_i^0)f_1(x,y)+[(x-x_i^0)^2+(y-y_i^0)^2]^{3/2}[A_1+\cdots]}{3[(x-x_i^0)^2+(y-y_i^0)^2]^{1/2}(y-y_i^0)f_1(x,y)+[(x-x_i^0)^2+(y-y_i^0)^2]^{3/2}[B_1+\cdots]},$$

and the slope of C_2 is

$$-\frac{\partial G_2/\partial x}{\partial G_2/\partial y}=\frac{3[(x-x_i^0)^2+(y-y_i^0)^2]^{1/2}(x-x_i^0)f_2(x,y)+[(x-x_i^0)^2+(y-y_i^0)^2]^{3/2}[B_2+\cdots]}{m_i y+3[(x-x_i^0)^2+(y-y_i^0)^2]^{1/2}(y-y_i^0)f_2(x,y+[(x-x_i^0)^2+(y-y_i^0)^2]^{3/2}[A_2+\cdots]}.$$

On evaluating these slopes at (x_i^0, y_i^0) we find that C_1 passes through P_i^0 tangent to $x=x_i^0$ and C_2 passes through P_i^0 tangent to $y=y_i^0$.

We shall determine next on which side of the tangent line the curve lies in the neighborhood of P_i^0. Let $x-x_i^0=r\cos\theta$, $y-y_i^0=r\sin\theta$ then equations (3.2) and (3.4) become, in polar coordinates,

$$\psi_1(r,\theta) = (m_i/r^2 + A_1 r)\cos\theta + B_1 r\sin\theta + \cdots = 0,$$
$$\psi_2(r,\theta) = (m_i/r^2 + A_2 r)\cos\theta + B_2 r\sin\theta + \cdots = 0.$$

From (3.3) we note that $\partial f_1/\partial y \equiv \partial f_2/\partial x$, and hence $B_1=A_2$. Since m_i and r are positive we may choose r so small that both (m_i/r^3+A_1) and (m_i/r^3+B_2) are positive. Then as θ varies from 0 to 2π, $\psi_1(r,\theta)$ changes sign at $\theta=\arctan[(m_i/r^3+A_1)/(-B_1)]$, and ψ_2 changes sign at

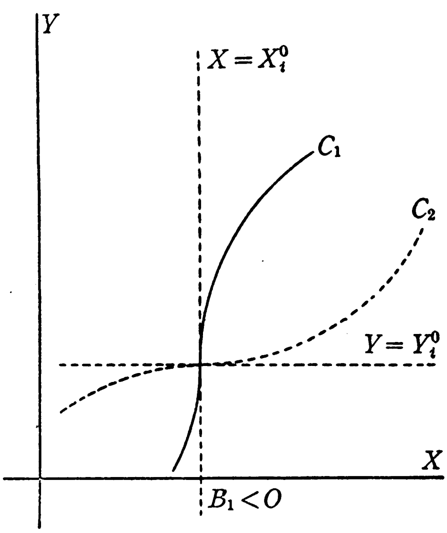

FIG. 1a

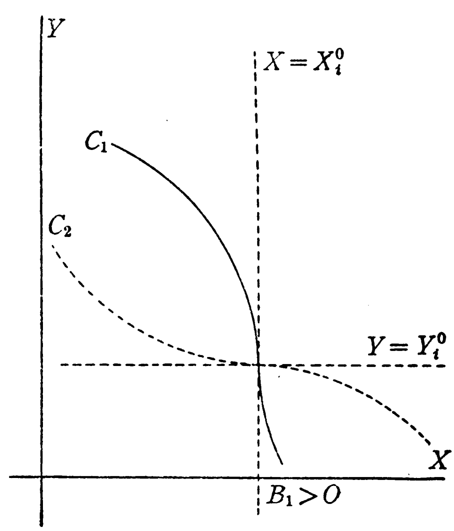

FIG. 1b

$\theta=\arctan[-B_1/(m_i/r^3+A_2)]$. Hence the curvature of both C_1 and C_2 at P_i^0 depends only upon the sign of B_1. The behavior of the curves at any P_i^0 is either that of Fig. 1a or Fig. 1b. In case $B_1=0$ the behavior of the curves is governed by the sign of the first nonzero quantity in the sequence

$$\partial^2 f_1/\partial y^2 \equiv \partial^2 f_2/\partial x\partial y,\ \partial^3 f_1/\partial y^3 \equiv \partial^3 f_2/\partial y^2\partial x,\ \cdots,$$

and the cases pictured in Figures 1a and 1b are all that can occur. In order to prove that there is at least one real solution of (3.2) other than the P_i^0, let us denote by P^0 that P_i^0 which has the largest ordinate. In case B_1 is negative at P^0 the curves are as shown in Fig. 2, and we shall prove that there is

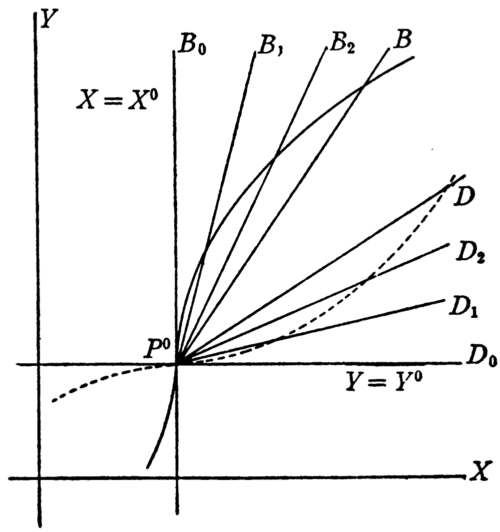

FIG. 2

at least one real intersection of C_1 and C_2 in that section of the plane for which $y>y^0$ and $x>x^0$.

By choice of notation there are no P_i^0 in this region and suppose that there is no other intersection. Let $P=(x, y)$ move from P^0 to $+\infty$ along $y=y^0$. Along this ray P meets C_2 first, namely at P^0, and then meets C_1 later, since ϕ_1 varies continuously from $+\infty$ to $-\infty$ as is easily verified in (3.1). Similarly, as P moves from P^0 to $+\infty$ along $x=x^0$ it meets C_1 first and C_2 later. Let this region of the plane be covered by rays leading from P^0 to ∞. As P moves from P^0 to ∞ along rays P^0D_1, P^0D_2, $\cdots$, it will meet C_2 first provided the slopes of the rays are sufficiently small, while along rays P^0B_1, P^0B_2, $\cdots$, it will meet C_1 first provided the slopes of these rays are sufficiently large. If there is no intersection of C_1 and C_2 in the region there is a last ray P^0D of the rays P^0D_i along which P meets C_2 first and a last ray P^0B of rays P^0B_i along which P meets C_1 first. The areas covered by the rays P^0D_i and P^0B_i may (a) overlap, (b) be adjacent, in which case P^0D and P^0B coincide, (c) be separated by a sector BP^0D.

Before taking up these three cases let us show that all rays from P^0 to ∞ that lie in the region $x>x^0$ and $y>y^0$ cut both curves at least once. From (3.1) it is seen that ϕ_1 and ϕ_2 are continuous functions of x and y for P within this region, and that ϕ_1 and ϕ_2 each vary continuously from $+\infty$ to $-\infty$ as P moves from P^0 to ∞ along any ray from P that lies in the region. Applying this result (a) to rays in the sector common to B_0P^0B and D_0P^0D, we conclude that every ray drawn from P^0 and lying in this sector, since it meets both curves and meets each curve first, must pass through an intersection of C_1 and C_2; (b) to the single ray common to both sectors, we reach the same conclusion as in (a); (c) to rays in the sector BP^0D separating those already covered, we find that every ray, since it does not meet C_1 first and does not meet C_2 first but does meet them both, must pass through an intersection of C_1 and C_2. In all cases one is led to a contradiction of the hypothesis that there is no intersection.

It is evident that the same proof can be applied in case B_1 is positive at P^0. In case there are several such points P^0 with the same ordinate the proof can be applied at any one of them yielding an intersection with ordinate greater than y^0. Thus we have the following theorem.

THEOREM 3.1. *There exists at least one point of libration at which a zero mass m_n can be placed, so that it will, together with the given $n-1$ positive masses, form a permanent configuration.*

4. **The solution as m_n becomes positive.** As shown in the last section there is a solution of (2.1) for positive λ, m_1, m_2, $\cdots$, m_{n-1}, and $m_n=0$. Let this solution be P_i^0. Since equations (2.1) are algebraic equations with coefficients functions of m_j, the solution functions x_i^0 and y_i^0 are continuous functions of m_n as long as the roots are finite and the equations do not have indeterminate forms. Consequently, x_i^0 and y_i^0 are continuous functions of m_n if no x_i^0 or no y_i^0 become infinite, or if no pair (x_i^0, y_i^0) becomes equal to another pair (x_k^0, y_k^0). Furthermore, the real roots x_i^0 and y_i^0 of algebraic equations (2.1) with real coefficients λ and m_i can disappear only by passing to infinity, or by an even number of real roots becoming complex conjugate quantities in pairs. Therefore, the problem is to determine as m_n varies whether

(1°) for all finite x_i^0 and y_i^0 any pair (x_j^0, y_j^0) can become equal to another pair (x_k^0, y_k^0),

(2°) any x_i^0 or y_i^0 can become infinite,

(3°) the solution (x_i^0, y_i^0) can ever become imaginary.

5. **The P_i^0 remain distinct.** This section is devoted to problem (1°) of the last section. We shall treat the more general case in which any m_j may vary.

Let the notation be chosen so that

$$x_1^0 \leqq x_2^0 \leqq x_3^0 \leqq \cdots \leqq x_{n-1}^0 \leqq x_n^0, \tag{5.1}$$

where there is at least one inequality in the noncollinear case. Suppose that as m_j varies all P_i^0 remain finite, and P_j^0 approaches P_k^0 in a manner (a) such that $x_j^0 \neq x_k^0$, and that $j<k$. In the jth equation (2.1) there is a term involving $(x_j^0 - x_k^0)/(r_{jk}^0)^3 = \alpha_{jk}^0$ which becomes negatively infinite. If this jth equation is satisfied another term $\alpha_{jl}^0 m_l$ must become positively infinite, and according to our notation $l<j$. Now α_{jl}^0 appears besides only in the lth equation and becomes negatively infinite in the term $\alpha_{lj}^0 m_j$. If the lth equation is satisfied a term $\alpha_{lp}^0 m_p$ must become positively infinite and according to the notation $p<l$. Continuing in this manner one arrives eventually at the situation in which a term involving α_{ts}^0 with one of the subscripts 1 becomes negatively infinite. But all terms in the first equation are negative except $-\lambda x_1^0$, which cannot become positively infinite under the hypothesis that the P_i^0 remain finite. Hence, the hypothesis that P_j^0 approaches P_k^0 in the manner (a) is false.

Suppose that P_j^0 approaches P_k^0 in the manner (b) such that $x_j^0 = x_k^0$. For this case let us choose the notation so that

$$y_1^0 \leqq y_2^0 \leqq y_3^0 \leqq \cdots \leqq y_{n-1}^0 \leqq y_n^0, \tag{5.2}$$

where again there is at least one inequality. In this notation, if P_j^0 becomes P_p^0 and P_k^0 becomes P_q^0, our hypothesis is that P_p^0 approaches P_q^0 in the manner $x_p^0 = x_q^0$ and $p<q$. There is a term in the $(n+p)$th equation (2.1) involving $(y_p^0 - y_q^0)/(r_{pq}^0)^3 = \beta_{pq}^0$, which becomes negatively infinite. By an argument similar to that of the last paragraph one eventually arrives at the situation in which a term involving β_{ts}^0 with one of the subscripts 1 becomes negatively infinite. But all terms in the $(n+1)$th equation are negative except $-\lambda y_1^0$, which cannot become positively infinite under the hypothesis that the P_i^0 remain finite. Hence, the hypothesis that P_j^0 approaches P_k^0 in the manner (b) is false. This completes the proof of the following theorem.

THEOREM 5.1. *As the m_i vary and the solution functions x_i^0 and y_i^0 remain finite the P_i^0 remain distinct.*

6. **The P_i^0 remain finite.** In order to show that as any m_j varies no x_i^0 or y_i^0 can become infinite, let us again adopt the notation (5.1) and suppose that x_j^0 becomes positively infinite. From the center of gravity equation

$$\sum_{i=1}^{n} m_i x_i^0 = 0, \tag{6.1}$$

some x_k^0 must become negatively infinite, and from (5.1) it follows that x_1^0 approaches $-\infty$ and x_n^0 approaches $+\infty$. Consider the first equation (2.1). In order that it be satisfied P_2^0 must approach P_1^0 in a manner such that $x_1^0 \neq x_2^0$, that is, x_2^0 must approach $-\infty$. Now consider the second equation. If it is to be satisfied, P_3^0 must approach P_2^0 in a manner such that $x_2^0 \neq x_3^0$, that is, x_3^0 must approach $-\infty$. Continuing this process we are led to the conclusion that all x_i^0 approach $-\infty$, hence the hypothesis must be false.

In order to show that no y_i^0 can become infinite, we adopt the notation (5.2) and suppose that y_i^0 becomes positively infinite as m_j varies. From the center of gravity equation

$$\sum_{i=1}^{n} m_i y_i^0 = 0, \tag{6.2}$$

and the notation (5.2), it follows that y_1^0 approaches $-\infty$, and y_n^0 approaches $+\infty$. Consider the $(n+1)$th, $(n+2)$th, and so on of equations (2.1) and the proof is the same as that in the preceding paragraph with the corresponding changes in notation. Thus we have the following theorem.

THEOREM 6.1. *As the m_i vary the P_i^0 remain finite.*

7. **The solution as m_r vanishes.** Before we take up the question of the solution becoming imaginary (3° of §4), it is to be shown that as m_n approaches zero the equations (2.1) and the solution functions x_i^0 and y_i^0 remain regular, and that there is, accordingly, only one limiting position of the P_i^0 for the value $m_n=0$. Let a solution for all positive m_i and λ be P_i', and let some m_j approach zero.

First, if some P_j' approaches some P_k' in any manner then it must be that P_j' approaches P_{j-1}' or P_{j+1}', for all other possibilities lead to the fact that one of the equations cannot be satisfied by arguments used in §§5 and 6.

Secondly, if P_j' approaches P_{j+1}' in the manner $x_j' = x_{j+1}'$, and if the notation is that of (5.2) the $(n+j)$th equation (2.1) cannot be satisfied unless P_{j-1}' approaches P_j' in the manner $x_{j-1}'=x_j'$. Thus P_{j-1}' approaches P_{j+1}' in the manner $x_{j-1}'=x_{j+1}'$ as m_j vanishes. But then the $(n+j+1)$th and $(n+j-1)$th equations cannot be satisfied unless P_{j-2}' approaches P_{j-1}' in the manner $x_{j-2}'=x_{j-1}'$ and P_{j+2}' approaches P_{j+1}' in the manner $x_{j+2}'=x_{j+1}'$. This shifts the difficulty to the $(n+j+2)$th and $(n+j-2)$th equations. On continuing this process one arrives at the $(n+1)$th and $(2n)$th equations which cannot be satisfied under the hypothesis it was necessary to make.

In case P_j' approaches P_{j+1}' in the manner $x_j' \neq x_{j+1}'$, then, if the notation is that of (5.1), the jth equation (2.1) cannot be satisfied unless P_{j-1}' approaches P_j' in the manner $x_{j-1}' \neq x_j'$ as m_j vanishes. Thus P_{j-1}' approaches P_{j+1}' in the manner $x_{j-1}' \neq x_{j+1}'$. But the $(j+1)$th and $(j-1)$th equations cannot be satisfied unless P_{j-2}' approaches P_{j-1}' in the manner $x_{j-2}' \neq x_{j-1}'$, and P_{j+2}' approaches P_{j+1}' in the manner $x_{j+2}' \neq x_{j+1}'$. This shifts the difficulty to the $(j-2)$th and $(j+2)$th equations, and on continuing this process one eventually arrives at the 1st and nth equations which cannot be satisfied under the hypothesis it was necessary to make.

In the third place, if any x_i' or y_i' becomes positively infinite as m_j vanishes, it must be that x_j' or y_j' becomes positively infinite, for all other possibilities lead to the fact that all x_i' or all y_i' become negatively infinite by arguments used in §6. If the notation is that of (5.1), the jth equation cannot be satisfied as x_j' becomes positively infinite unless P_j' approaches P_{j-1}' in the manner $x_j' \neq x_{j-1}'$, that is, x_{j-1}' must approach $+\infty$. Then the $(j-1)$th equation cannot be satisfied unless P_{j-1}' approaches P_{j-2}' in the manner $x_{j-1}' \neq x_{j-2}'$, in which case x_{j-2}' approaches $+\infty$. Continuing this process one arrives at the conclusion that $x_1', x_2', \cdots, x_{j-1}', x_j'$ become positively infinite, and by choice of notation $x_{j+1}', x_{j+2}', \cdots, x_n'$ become positively infinite. Hence the center of gravity equation (6.1) cannot be satisfied, and the hypothesis that x_j' become positively infinite is false.

The proof that y_j' cannot become positively infinite as m_j vanishes is precisely the same as that of the last paragraph with the notation that of (5.2) and the argument beginning with the consideration of the $(n+j)$th equation (2.1). The proof that no x_j' or y_j' can become negatively infinite as m_j vanishes will read the same as the proof above with the words "positively infinite" replaced by "negatively infinite." This concludes the proof of the following:

THEOREM 7.1. *Any solution P_i' of* (2.1) *for positive m_i and λ remains regular as any m_i vanishes.*

In particular we shall use the following corollary.

COROLLARY 7.1. *The solution P_i' for positive m_i and λ remains regular as m_n vanishes.*

8. **The condition that the P_i^0 remain real.** Since the solutions of (2.1) are continuous functions of m_n, it follows that no two solutions which are real for $m_n=0$ can ever become complex conjugate solutions for any positive value of m_n without having first become equal. If a multiple solution of (2.1) is impossible for a set of finite positive values of m_i, then it is impossible for any real solution to disappear by becoming complex or for any complex solution to become real.

The conditions that a set of simultaneous algebraic equations shall have a multiple solution are that a set of values of the variable shall satisfy the equations, and that the Jacobian of the functions with respect to the independent variables shall vanish for the same set of values.

Since equations (2.1) are invariant under a rotation of axes, let the axes be chosen so that $y_2=0$ in case $n=2$, and so that $y_1=0$ for $n\geqq 3$. It is to be shown that for $n=2$ the first three equations (2.1) are independent for all positive values of λ, m_1 and m_2, while for $n\geqq 3$ the $2n-1$ equations obtained from (2.1) by omitting the $(n+1)$th form an independent set for any positive λ, m_1, m_2, and sufficiently small positive $m_3, m_4, \cdots, m_n$.

Let F_i and F_{n+i} denote the left members of (2.1) and let the Jacobian of these functions be Δ_n. Furthermore, let the minor obtained by deleting the $(n+1)$th row and $(n+1)$th column of Δ_n be D_n and let $n\geqq 3$. A real solution P_i^0 of (2.1) for positive $\lambda, m_1, m_2, \cdots, m_{n-1}$, and $m_n=0$ will remain real as m_n

increases from zero through positive values for which D_n does not vanish. This will be established when it is shown that $D_n \neq 0$ for sufficiently small positive m_j, $j=3, 4, \cdots, n$.

The following notation will be used. $\partial F_i/\partial x_j = a_{ij}$, $\partial F_{n+i}/\partial y_j = b_{ij}$, $\partial F_i/\partial y_j = \partial F_{n+i}/\partial x_j = c_{ij}$. From (2.1) it follows that, for $i \neq j$,

$$a_{ij} = \frac{3(x_i - x_j)^2 - (r_{ij})^2}{(r_{ij})^5} m_j, \qquad b_{ij} = \frac{3(y_i - y_j)^2 - (r_{ij})^2}{(r_{ij})^5} m_j, \tag{8.1}$$
$$c_{ij} = \frac{3(x_i - x_j)(y_i - y_j)}{(r_{ij})^5} m_j,$$

and, for $i=j$,

$$a_{ii} = -\sum_{k=1}^{n-1} a_{i\,i+k} - \lambda, \quad b_{ii} = -\sum_{k=1}^{n-1} b_{i\,i+k} - \lambda, \quad c_{ii} = -\sum_{k=1}^{n-1} c_{i\,i+k}, \tag{8.2}$$

where $a_{il+n} = a_{il}$, $b_{il+n} = b_{il}$, $c_{il+n} = c_{il}$.

9. **Case** $n=2$. Equations (2.1) in this case consist of four equations. Let us choose the axes so that $y_2=0$, then the solution of these equations is

$$x_1 = -m_2/(\lambda M^2)^{1/3}, \qquad y_1 = 0, \tag{9.1}$$
$$x_2 = m_1/(\lambda M^2)^{1/3}, \qquad y_2 = 0,$$

where $M = m_1 + m_2$.

To compute the elements of Δ_2 the following quantities, $x_1 - x_2 = -(M/\lambda)^{1/3}$, $y_1 - y_2 = 0$, $r_{12} = (M/\lambda)^{1/3}$, are substituted in (8.1) and (8.2). We find

$$\Delta_2 = \begin{vmatrix} -2\lambda m_2/M - \lambda & 2\lambda m_2/M & 0 & 0 \\ 2\lambda m_1/M & -2\lambda m_1/M - \lambda & 0 & 0 \\ 0 & 0 & -\lambda m_1/M & -\lambda m_2/M \\ 0 & 0 & -\lambda m_1/M & -\lambda m_2/M \end{vmatrix}.$$

It is quite evident that $\Delta_2 = 0$ and it remains to be shown that the minor D_2 in the upper left-hand corner is different from zero. After adding the second column to the first, and then subtracting the first row from the second,

$$D_2 = \begin{vmatrix} -\lambda & 2\lambda m_2/M & 0 \\ 0 & -3\lambda & 0 \\ 0 & 0 & -\lambda m_1/M \end{vmatrix}.$$

Since $D_2 \neq 0$ for all finite positive values of λ, m_1 and m_2 the solution of (2.1) for $m_2=0$, namely, $x_1=0$, $y_1=0$, $x_2=(m_1/\lambda)^{1/3}$, $y_2=0$, remains real and varies continuously to the values given in (9.1) for all finite positive values of λ, m_1 and m_2.

10. **Case** $n=3$. By choosing the axes so that $y_1=0$ the solution of the six equations (2.1) is

$$(10.1)\quad \begin{aligned} x_1 &= 2(m_2^2 + m_2m_3 + m_3^2)K, & y_1 &= 0,\\ x_2 &= -(m_2m_3 + m_1m_3 + 2m_1m_2 - m_3^2)K, & y_2 &= -3^{1/2}M\,m_3K,\\ x_3 &= -(2m_1m_3 + m_1m_2 + m_2m_3 - m_2^2)K, & y_3 &= 3^{1/2}M\,m_2K, \end{aligned}$$

where $M = m_1 + m_2 + m_3$, and $K = 1/(2\lambda^{1/3}M^{2/3}(m_2^2 + m_2m_3 + m_3^2)^{1/2})$.

If the elements of Δ_3 are computed in terms of λ, m_1, m_2 and m_3, Δ_3 will vanish, as can be shown by direct computation, since the system (2.1) is dependent and y_1 can be chosen arbitrarily. D_3 is the minor obtained by leaving out the 4th row and 4th column of Δ_3 and has determinant different from zero for all finite positive λ, m_1, m_2, and m_3, as a rather long series of computations will show.

However, the proof that D_3 is different from zero for all λ, m_1, m_2 and sufficiently small m_3 is short and will be given here, since it is the same type of proof that will be used in the general case. In the notation (8.1) and (8.2),

$$(10.2)\quad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & c_{11} & c_{12} & c_{13}\\ a_{21} & a_{22} & a_{23} & c_{21} & c_{22} & c_{23}\\ a_{31} & a_{32} & a_{33} & c_{31} & c_{32} & c_{33}\\ c_{11} & c_{12} & c_{13} & b_{11} & b_{12} & b_{13}\\ c_{21} & c_{22} & c_{23} & b_{21} & b_{22} & b_{23}\\ c_{31} & c_{32} & c_{33} & b_{31} & b_{32} & b_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & c_{12} & c_{13}\\ a_{21} & a_{22} & a_{23} & c_{22} & c_{23}\\ a_{31} & a_{32} & a_{33} & c_{32} & c_{33}\\ c_{21} & c_{22} & c_{23} & b_{22} & b_{23}\\ c_{31} & c_{32} & c_{33} & b_{32} & b_{33} \end{vmatrix}.$$

After rearranging rows and columns in Δ_3 we have

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & c_{11} & c_{12} & a_{13} & c_{13}\\ a_{21} & a_{22} & c_{21} & c_{22} & a_{23} & c_{23}\\ c_{11} & c_{12} & b_{11} & b_{12} & c_{13} & b_{13}\\ c_{21} & c_{22} & b_{21} & b_{22} & c_{23} & b_{23}\\ a_{31} & a_{32} & c_{31} & c_{32} & a_{33} & c_{33}\\ c_{31} & c_{32} & b_{31} & b_{32} & c_{33} & b_{33} \end{vmatrix},$$

and if $m_3 = 0$, Δ_3 and D_3 become

$$\Delta_3 = \left|\begin{array}{cccc:cc} & & & & 0 & 0\\ & \Delta_2 & & & 0 & 0\\ & & & & 0 & 0\\ & & & & 0 & 0\\ \hdashline a_{31} & a_{32} & c_{31} & c_{32} & a_{33} & c_{33}\\ c_{31} & c_{32} & b_{31} & b_{32} & c_{33} & b_{33} \end{array}\right|,$$

and

$$(10.3)\quad D_3 = \begin{vmatrix} -\left(\frac{2m_2}{M}+1\right)\lambda & \frac{2m_2}{M}\lambda & 0 & 0 & 0 \\ \frac{2m_1}{M}\lambda & -\left(\frac{2m_1}{M}+1\right)\lambda & 0 & 0 & 0 \\ -\frac{m_1}{4M}\lambda & -\frac{m_2}{M}\lambda & -\frac{3}{4}\lambda & \frac{-3^{3/2}m_2}{M}\lambda & \frac{-3^{3/2}(m_1-m_2)}{M}\lambda \\ 0 & 0 & 0 & -\frac{m_2}{M}\lambda & 0 \\ \frac{3^{3/2}m_1}{M}\lambda & \frac{-3^{3/2}m_2}{M}\lambda & \frac{-3^{3/2}(m_1-m_2)}{M}\lambda & \frac{5m_2}{4M}\lambda & -\frac{9}{4}\lambda \end{vmatrix}.$$

On expanding (10.3) we obtain

$$(10.4)\qquad D_3 = -(3^4 m_1 m_2^2/4M^3)\lambda^5.$$

Since D_3 of (10.2) is a continuous function of m_3 and, as (10.4) reveals, is different from zero for positive λ, m_1, m_2, and $m_3=0$, it is different from zero for sufficiently small positive m_3.

11. **Case** $n=4$. Using the notation (8.1) and (8.2) we have

$$\Delta_4 = \left|\begin{array}{c:c} a_{ij} & c_{ij} \\ \hdashline c_{ij} & b_{ij} \end{array}\right|, \qquad i, j = 1, 2, 3, 4,$$

and

$$D_4 = \left|\begin{array}{cccc:ccc} & & & & c_{12} & c_{13} & c_{14} \\ & & a_{ij} & & c_{22} & c_{23} & c_{24} \\ & & & & c_{32} & c_{33} & c_{34} \\ & & & & c_{42} & c_{43} & c_{44} \\ \hdashline c_{21} & c_{22} & c_{23} & c_{24} & b_{22} & b_{23} & b_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} & b_{32} & b_{33} & b_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} & b_{42} & b_{43} & b_{44} \end{array}\right|, \qquad i, j = 1, 2, 3, 4.$$

After rearranging rows and columns, we have

$$\Delta_4 = \left|\begin{array}{cccccc:cc} & & & & & & a_{14} & c_{14} \\ & & & & & & a_{24} & c_{24} \\ & & & \Delta_3 & & & a_{34} & c_{34} \\ & & & & & & c_{14} & b_{14} \\ & & & & & & c_{24} & b_{24} \\ & & & & & & c_{34} & b_{34} \\ \hdashline a_{41} & a_{42} & a_{43} & c_{41} & c_{42} & c_{43} & a_{44} & c_{44} \\ c_{41} & c_{42} & c_{43} & b_{41} & b_{42} & b_{43} & c_{44} & b_{44} \end{array}\right|, \quad D_4 = \left|\begin{array}{ccccc:cc} & & & & & a_{14} & c_{14} \\ & & & & & a_{24} & c_{24} \\ & & D_3 & & & a_{34} & c_{34} \\ & & & & & c_{24} & b_{24} \\ & & & & & c_{34} & b_{34} \\ \hdashline a_{41} & a_{42} & a_{43} & c_{42} & c_{43} & a_{44} & c_{44} \\ c_{41} & c_{42} & c_{43} & b_{42} & b_{43} & c_{44} & b_{44} \end{array}\right|.$$

For $m_4=0$ these become

$$\Delta_4 = \left|\begin{array}{cccccc:cc} & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & \Delta_3 & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ \hdashline a_{41} & a_{42} & a_{43} & c_{41} & c_{42} & c_{43} & a_{44} & c_{44} \\ c_{41} & c_{42} & c_{43} & b_{41} & b_{42} & b_{43} & c_{44} & b_{44} \end{array}\right|, \quad D_4 = \left|\begin{array}{ccccc:cc} & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & D_3 & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ \hdashline a_{41} & a_{42} & a_{43} & c_{42} & c_{43} & a_{44} & c_{44} \\ c_{41} & c_{42} & c_{43} & b_{42} & b_{43} & c_{44} & b_{44} \end{array}\right|.$$

Since x_i and y_i are continuous functions of m_4, D_4 is a continuous function of m_4 and is different from zero for $m_4=0$, provided that λ, m_1 and m_2 are positive, that m_3 is so chosen that $D_3 \neq 0$, and that (x_4, y_4) is not a multiple solution of the 4th and 8th equations (2.1). Hence, $D_4 \neq 0$ for sufficiently small positive values of m_4, and the solution, real for positive λ, m_1, m_2, m_3, and $m_4=0$, varies continuously and remains real for sufficiently small positive values of m_4.

12. **Existence of a solution for positive m_n.** Continuing the process of §§9, 10, 11 it is possible for one to choose $m_5, m_6, \cdots, m_{n-1}$ successively so that $D_5, D_6, \cdots, D_{n-1}$ remain different from zero. In the general case Δ_n will be zero due to the dependence of equations (2.1). The minor D_n obtained from Δ_n by deleting the $(n+1)$th row and $(n+1)$th column will be, in the notation of (8.1) and (8.2) and after rows and columns have been rearranged,

$$D_n = \left|\begin{array}{cccccccc:cc} & & & & & & & & a_{1n} & c_{1n} \\ & & & & & & & & a_{2n} & c_{2n} \\ & & & & & & & & \vdots & \vdots \\ & & & & & & & & a_{n-1\,n} & c_{n-1\,n} \\ & & & & D_{n-1} & & & & c_{2n} & b_{2n} \\ & & & & & & & & c_{3n} & b_{3n} \\ & & & & & & & & \vdots & \vdots \\ & & & & & & & & c_{n-1\,n} & b_{n-1\,n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{n\,n-1} & c_{n2} & c_{n3} & \cdots & c_{n\,n-1} & a_{nn} & c_{nn} \\ c_{n1} & c_{n2} & \cdots & c_{n\,n-1} & b_{n2} & b_{n3} & \cdots & b_{n\,n-1} & c_{nn} & b_{nn} \end{array}\right|.$$

For $m_n=0$ we have

$$D_n = D_{n-1} \left|\begin{array}{cc} \partial F_n/\partial x_n & \partial F_n/\partial y_n \\ \partial F_{2n}/\partial x_n & \partial F_{2n}/\partial y_n \end{array}\right|.$$

Since x_i and y_i are continuous functions of m_n, D_n is a continuous function of m_n; and, if (x_n, y_n) is not a multiple solution of the nth and $(2n)$th equations (2.1), $D_n \neq 0$ for $m_n = 0$. Therefore, $D_n \neq 0$ for sufficiently small values of m_n, and the solution, real for positive λ, m_1, m_2, $\cdots$, m_{n-1}, and $m_n = 0$, varies continuously and remains real for sufficiently small positive values of m_n. This concludes the proof of the following theorem.

THEOREM 12.1. *For arbitrary finite positive* λ, m_1, m_2, *and sufficiently small positive* m_3, m_4, $\cdots$, m_n *there exists at least one real solution of equations* (2.1).

It should be noted that, from the Theorem of Lagrange and our remark in §10, m_3 need not be restricted and hence we have the slightly more general theorem.

THEOREM 12.2. *For arbitrary finite positive masses* m_1, m_2, m_3, *arbitrary finite angular velocity, and sufficiently small positive masses* m_4, m_5, $\cdots$, m_n, *there exists at least one noncollinear plane permanent configuration.*

13. **A special form for the equations.** In §12 we showed that under certain conditions $D_n \neq 0$. It follows from this result that for any λ, m_1, m_2, and properly chosen m_3, m_4, $\cdots$, m_n, equations (2.1) with the $(n+1)$th deleted form an independent set. It is easy to show, by processes similar to those of §§8 to 12, that the first and $(n+2)$th may be replaced by the center of gravity equations (6.1) and (6.2) yielding an independent set of $2n-1$ equations equivalent to the one just mentioned.

It is our purpose in this section to put this latter set of $2n-1$ equations into a form convenient for calculating a solution. We shall then solve them for the coordinates in terms of λ and the masses m_i. Let the mass ratios be 1, $m_2/m_1 = \mu$, $m_{i+2}/m_1 = \sigma_i$ $(i = 1, 2, \cdots, n-2)$ and for the particular solution we wish to construct we shall take $\lambda = m_1$. The equations then become

$$
\begin{aligned}
&x_1 + \mu x_2 + \sum x_{2+j}\sigma_j = 0,\\
&y_1 + \mu y_2 + \sum y_{2+j}\sigma_j = 0,\\
&-x_2 + \frac{x_2 - x_1}{(r_{21})^3} + \sum \frac{x_2 - x_{2+j}}{(r_{2\,2+j})^3}\sigma_j = 0,\\
(13.1)\qquad &-x_i + \frac{x_i - x_1}{(r_{i1})^3} + \frac{x_i - x_2}{(r_{i2})^3}\mu + \sum{}^* \frac{x_i - x_{2+j}}{(r_{i\,2+j})^3}\sigma_j = 0,\\
&\hspace{20em} i = 3, 4, \cdots, n\\
&-y_i + \frac{y_i - y_1}{(r_{i1})^3} + \frac{y_i - y_2}{(r_{i2})^3}\mu + \sum{}^* \frac{y_i - y_{2+j}}{(r_{i\,2+j})^3}\sigma_j = 0,
\end{aligned}
$$

where $\sum$ denotes $\sum_{j=1}^{n-2}$ and $\sum^*$ denotes the sum over the same j except for $j = i-2$.

Let us multiply the next to last equation by y_i and the last by $-x_i$ and form the sum. The resulting equation,

$$\frac{x_i y_1 - x_1 y_i}{(r_{i1})^3} + \frac{x_i y_2 - x_2 y_i}{(r_{i2})^3}\mu + \sum{}^* \frac{x_i y_{2+j} - x_{2+j} v_i}{(r_{i\,2+j})^3}\sigma_j = 0,$$

is used to replace the next to last above. By means of the complex numbers

$$z_{ij} = (y_i - y_j + (-1)^{1/2}(x_i y_j - x_j y_i))/(r_{ij})^3$$

we are now able to write the $2n-1$ equations (13.1) in the form

$$\begin{aligned} &x_1 + \mu x_2 + \sum x_{2+j}\sigma_j = 0, \\ &y_1 + \mu y_2 + \sum y_{2+j}\sigma_j = 0, \\ &-x_2 + \frac{x_2 - x_1}{(r_{21})^3} + \sum \frac{x_2 - x_{2+j}}{(r_{2\,2+j})^3}\sigma_j = 0, \\ &-y_i + z_{i1} + z_{i2}\mu + \sum{}^* z_{i\,2+j}\sigma_j = 0, \qquad i = 3, 4, \cdots, n, \end{aligned} \tag{13.2}$$

which is convenient for computing the solution.

In the case $n=4$ the coordinates may be expressed as power series in the mass ratios σ_1 and σ_2 with coefficients which are Laurent series in μ. For $n>4$ the coordinates may be expressed as power series in $(\sigma_i/3)^{1/3}$ with coefficients Laurent series in μ.

14. **Solution for $n=4$.** In this case the seven equations (13.2) are

$$\begin{aligned} &x_1 + \mu x_2 + \sigma_1 x_3 + \sigma_2 x_4 = 0, \\ &y_1 + \mu y_2 + \sigma_1 y_3 + \sigma_2 y_4 = 0, \\ &-x_2 + \frac{x_2 - x_1}{(r_{21})^3} + \frac{x_2 - x_3}{(r_{23})^3}\sigma_1 + \frac{x_2 - x_4}{(r_{24})^3}\sigma_2 = 0, \\ &\frac{x_3 y_1 - x_1 y_3}{(r_{31})^3} + \frac{x_3 y_2 - x_2 y_3}{(r_{32})^3}\mu + \frac{x_3 y_4 - x_4 y_3}{(r_{34})^3}\sigma_2 = 0, \\ &-y_3 + \frac{y_3 - y_1}{(r_{31})^3} + \frac{y_3 - y_2}{(r_{32})^3}\mu + \frac{y_3 - v_4}{(r_{34})^3}\sigma_2 = 0, \\ &\frac{x_4 y_1 - x_1 y_4}{(r_{41})^3} + \frac{x_4 y_2 - x_2 y_4}{(r_{42})^3}\mu + \frac{x_4 y_3 - x_3 y_4}{(r_{43})^3}\sigma_1 = 0, \\ &-y_4 + \frac{y_4 - y_1}{(r_{41})^3} + \frac{y_4 - y_2}{(r_{42})^3}\mu + \frac{y_4 - y_3}{(r_{43})^3}\sigma_1 = 0. \end{aligned} \tag{14.1}$$

Since one of the coordinates is arbitrary we may choose $y_1=0$. The solution of these equations may be obtained as follows. With $\sigma_2 \doteq 0$ the first five equations (14.1) form an independent set of the six equations (2.1), and the solution functions may be obtained from (10.1) by proper change of parameters. The solution functions thus obtained and arranged in powers of σ_1 are

$$
\begin{aligned}
x_1 &= \mu(1+\mu)^{-2/3} + \frac{3-\mu}{6}(1+\mu)^{-5/3}\sigma_1 + \cdots, \\
x_2 &= -(1+\mu)^{-2/3} + \frac{1-3\mu}{6}(1+\mu)^{-5/3}\sigma_1 + \cdots, \\
y_2 &= \frac{3^{1/2}}{2}\left[0 - \frac{(1+\mu)^{1/3}}{\mu}\sigma_1 + \cdots\right], \qquad (14.2)\\
x_3 &= -\frac{1-\mu}{2}(1+\mu)^{-2/3} - \frac{9+14\mu+13\mu^2}{12\mu}(1+\mu)^{-5/3}\sigma_1 + \cdots, \\
y_3 &= \frac{3^{1/2}}{2}\left[(1+\mu)^{1/3} - \frac{3+\mu}{6\mu}(1+\mu)^{-2/3}\sigma_1 + \cdots\right].
\end{aligned}
$$

The last two equations (14.1) may be obtained from the preceding two by the transformation, in two rowed notation,

$$
(14.3) \qquad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & y_2 & y_3 & y_4 & \mu & \sigma_1 & \sigma_2 \\ x_1 & x_2 & x_4 & x_3 & -y_2 & -y_4 & -y_3 & \mu & \sigma_2 & \sigma_1 \end{pmatrix},
$$

and then setting $\sigma_1=0$. Hence the solution functions x_4, y_4 of the last two equations may be obtained from the solution functions x_3, y_3 except for the coefficients of the powers of σ_1. On substituting

$$
\begin{aligned}
x_4 &= -((1-\mu)/2)(1+\mu)^{-2/3} + c_1(\mu)\sigma_1 + \cdots, \\
y_4 &= -(3^{1/2}/2)\left[(1+\mu)^{1/3} + c_2(\mu)\sigma_1 + \cdots\right]
\end{aligned}
$$

in the last two equations (14.1) and equating coefficients we find

$$
(14.4) \qquad
\begin{aligned}
c_1(\mu) &= \left(\frac{3^{1/2}}{27} + \frac{5}{12}\right)\frac{1}{\mu} - \left(\frac{2(3)^{1/2}}{81} - \frac{29}{36}\right) \\
&\quad - \left(\frac{3^{1/2}}{81} + \frac{7}{9}\right)\mu + \cdots, \\
c_2(\mu) &= \left(\frac{2(3)^{1/2}}{81} + \frac{5}{18}\right)\frac{1}{\mu} + \left(\frac{8(3)^{1/2}}{243} + \frac{11}{54}\right) \\
&\quad + \left(\frac{2(3)^{1/2}}{81} - \frac{8}{27}\right)\mu + \cdots.
\end{aligned}
$$

This completes the solution of (14.1) as functions of μ and σ_1.

In order to obtain the coefficients of σ_2 in the solution functions of (14.1) we may use the fact that the system (14.1) remains unchanged by the transformation (14.3) and that the solution functions must have this same property. Coefficients of higher powers of the parameters may be computed, and herein lies the convenience of the form (14.1). The coefficients of any term $\sigma_1^p\sigma_2^q$ in the expansion of $x_1, y_2, x_2, x_3, y_3, x_4, y_4$ may be determined successively, and in this order, by substituting series with undetermined coefficients into the equations in the order given in (14.1).

The solution for $n=4$, computed as described above and arranged in powers of σ_2, will follow immediately. The coefficients in this series are themselves arranged in powers of σ_1 with coefficients which are Laurent series in μ.

$$
\begin{aligned}
x_1 &= \mu(1+\mu)^{-2/3} + \frac{3-\mu}{6}(1+\mu)^{-5/3}\sigma_1 + \cdots + \left\{\frac{3-\mu}{6}(1+\mu)^{-5/3}\right.\\
&\quad \left. + \left[-\left(\frac{2(3)^{1/2}}{27}+\frac{1}{12}\right)\frac{1}{\mu}+\cdots\right]\sigma_1+\cdots\right\}\sigma_2+\cdots,\\
x_2 &= -(1+\mu)^{-2/3} + \frac{1-3\mu}{6}(1+\mu)^{-5/3}\sigma_1 + \cdots + \left\{\frac{1-3\mu}{6}(1+\mu)^{-5/3}\right.\\
&\quad \left. + \left[-\frac{3}{4\mu^2}+\cdots\right]\sigma_1+\cdots\right\}\sigma_2+\cdots,\\
y_2 &= \frac{3^{1/2}}{2}\left(0 - \frac{(1+\mu)^{1/3}}{\mu}\sigma_1 + \cdots + \left\{\frac{(1+\mu)^{1/3}}{\mu}\right.\right.\\
&\quad \left.\left. + \left[\frac{0}{\mu^2}+\cdots\right]\sigma_1+\cdots\right\}\sigma_2+\cdots\right),\\
x_3 &= -\frac{1-\mu}{2}(1+\mu)^{-2/3} - \frac{9+14\mu+13\mu^2}{12\mu}(1+\mu)^{-5/3}\sigma_1+\cdots \\
&\quad + \left\{\left(\frac{3^{1/2}}{27}+\frac{5}{12}\right)\frac{1}{\mu} - \left(\frac{2(3)^{1/2}}{81}-\frac{29}{36}\right) - \left(\frac{3^{1/2}}{81}+\frac{7}{9}\right)\mu\right.\\
&\quad \left. + \cdots + \left[\left(\frac{5(3)^{1/2}}{2\cdot 3^5}-\frac{1687}{2\cdot 3^6}\right)\frac{1}{\mu^2}+\cdots\right]\sigma_1+\cdots\right\}\sigma_2+\cdots,\\
y_3 &= \frac{3^{1/2}}{2}\left((1+\mu)^{1/3} - \frac{3+\mu}{6\mu}(1+\mu)^{-2/3}\sigma_1+\cdots\right.\\
&\quad + \left\{\left(\frac{2(3)^{1/2}}{81}+\frac{5}{18}\right)\frac{1}{\mu} + \left(\frac{8(3)^{1/2}}{243}+\frac{11}{54}\right)\right.\\
&\quad + \left(\frac{2(3)^{1/2}}{81}-\frac{8}{27}\right)\mu+\cdots\\
&\quad \left.\left. + \left[\left(\frac{41(3)^{1/2}}{3^6}+\frac{3173}{2^2 3^7}\right)\frac{1}{\mu^2}+\cdots\right]\sigma_1+\cdots\right\}\sigma_2+\cdots\right),
\end{aligned}
\tag{14.5}
$$

$$
\begin{aligned}
x_4 = & -\frac{1-\mu}{2}(1+\mu)^{-2/3} + \left[\left(\frac{3^{1/2}}{27}+\frac{5}{12}\right)\frac{1}{\mu} - \left(\frac{2(3)^{1/2}}{81}-\frac{29}{36}\right)\right. \\
& \left. - \left(\frac{3^{1/2}}{81}+\frac{7}{9}\right)\mu + \cdots\right]\sigma_1 + \cdots \\
& + \left\{ -\frac{9+14\mu+13\mu^2}{12\mu}(1+\mu)^{-5/3}\right. \\
& \left. + \left[\left(\frac{5(3)^{1/2}}{2\cdot 3^5} - \frac{1687}{2\cdot 3^6}\right)\frac{1}{\mu^2} + \cdots\right]\sigma_1 + \cdots \right\}\sigma_2 + \cdots, \qquad (14.5) \\
y_4 = & -\frac{3^{1/2}}{2}\left((1+\mu)^{1/3} + \left[\left(\frac{2(3)^{1/2}}{81}+\frac{5}{18}\right)\frac{1}{\mu} + \left(\frac{8(3)^{1/2}}{243}+\frac{11}{54}\right)\right.\right. \\
& \left. + \left(\frac{2(3)^{1/2}}{81}-\frac{8}{27}\right)\mu + \cdots\right]\sigma_1 + \cdots + \left\{-\frac{3+\mu}{6\mu}(1+\mu)^{-2/3}\right. \\
& \left.\left. + \left[\left(\frac{41(3)^{1/2}}{3^6}+\frac{3173}{2^2\cdot 3^7}\right)\frac{1}{\mu^2} + \cdots\right]\sigma_1 + \cdots\right\}\sigma_2 + \cdots\right).
\end{aligned}
$$

15. **Points of libration for $n=4$, 5.** In order to find a point where $m_5=0$ may be introduced, we must solve the pair of equations given by the last of equations (13.2) for the case $n=5$, namely,

$$
\begin{aligned}
& \frac{A_{51}}{(r_{51})^3} + \frac{A_{52}}{(r_{52})^3}\mu + \frac{A_{53}}{(r_{53})^3}\sigma_1 + \frac{A_{54}}{(r_{54})^3}\sigma_2 = 0, \\
& -y_5 + \frac{y_5}{(r_{51})^3} + \frac{y_5-y_2}{(r_{52})^3}\mu + \frac{y_5-y_3}{(r_{53})^3}\sigma_1 + \frac{y_5-y_4}{(r_{54})^3}\sigma_2 = 0,
\end{aligned} \qquad (15.1)
$$

where $A_{ij}=x_iy_j-x_jy_i$.

If $\sigma_1=0$ these equations are precisely those which determine P_3, hence P_5 coincides with P_3 for $\sigma_1=0$. Furthermore, $P_5\equiv P_3$ is a solution of equations (15.1) for $\sigma_1\neq 0$, the identity being in μ, σ_1 and σ_2. In order to obtain a solution of (15.1) having the property (in §5) that $P_5\not\equiv P_3$, we let $\sigma_1=3\nu^3$. Upon substituting

$$
\begin{aligned}
x_5 = & -\frac{1-\mu}{2}(1+\mu)^{-2/3} + p_1(\mu)\nu + p_2(\mu)\nu^2 \\
& + p_3(\mu)\nu^3 + \cdots + c_1(\mu)\sigma_2 + \cdots, \\
y_5 = & \frac{3^{1/2}}{2}[(1+\mu)^{1/3} + q_1(\mu)\nu + q_2(\mu)\nu^2 \\
& + q_3(\mu)\nu^3 + \cdots + c_2(\mu)\sigma_2 + \cdots]
\end{aligned} \qquad (15.2)
$$

in (15.1) the $p_i(\mu)$ and $q_i(\mu)$ may be determined by equating coefficients of ν.

The functions $c_1(\mu)$ and $c_2(\mu)$ are the same as in (14.4). The result is

$$
\begin{aligned}
&p_1(\mu) = \pm\left(\frac{1}{2} - \frac{1}{4}\mu + \cdots\right), \quad p_2(\mu) = \pm\left(\frac{1}{6} + \frac{125}{144}\mu + \cdots\right), \\
&\qquad p_3(\mu) = -\frac{9 + 14\mu + 13\mu^2}{4\mu}(1+\mu)^{-5/3}, \\
(15.3)\quad &q_1(\mu) = \pm\left(-1 - \frac{1}{2}\mu + \cdots\right), \quad q_2(\mu) = \pm\left(\frac{1}{3} + \frac{25}{72}\mu + \cdots\right), \\
&\qquad q_3(\mu) = -\frac{3+\mu}{2\mu}(1+\mu)^{-2/3}.
\end{aligned}
$$

Thus, as σ_1 increases from zero to a positive value, a point of libration, denoted by P_5^*, branches off from P_3 and moves to a finite distance from P_3. We shall use only the upper sign in (15.3).

The solution of

$$
(15.4)\quad
\begin{aligned}
&\frac{A_{61}}{(r_{61})^3} + \frac{A_{62}}{(r_{62})^3}\mu + \frac{A_{63}}{(r_{63})^3}\sigma_1 + \frac{A_{64}}{(r_{64})^3}\sigma_2 = 0, \\
&-y_6 + \frac{y_6}{(r_{61})^3} + \frac{y_6 - y_2}{(r_{62})^3}\mu + \frac{y_6 - y_3}{(r_{63})^3}\sigma_1 + \frac{y_6 - y_4}{(r_{64})^3}\sigma_2 = 0
\end{aligned}
$$

for the libration point P_6^* may now be obtained from (15.2) by the transformation

$$
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & y_2 & y_3 & y_4 & y_5 & y_6 & \mu & \sigma_1 & \sigma_2 \\
x_1 & x_2 & x_4 & x_3 & x_6 & x_5 & -y_2 & -y_4 & -y_3 & -y_6 & -y_5 & \mu & \sigma_2 & \sigma_1
\end{pmatrix},
$$

which reduces (15.4) to (15.1). Therefore, as σ_2 increases from zero to a positive value, a point of libration P_6^*, whose coordinates are

$$
(15.5)\quad
\begin{aligned}
x_6 = &-\frac{1-\mu}{2}(1+\mu)^{-2/3} + c_1(\mu)\sigma_1 + \cdots + p_1(\mu)\left(\frac{\sigma_2}{3}\right)^{1/3} \\
&+ p_2(\mu)\left(\frac{\sigma_2}{3}\right)^{2/3} + p_3(\mu)\left(\frac{\sigma_2}{3}\right) + \cdots, \\
y_6 = &-\frac{3^{1/2}}{2}\left[(1+\mu)^{1/3} + c_2(\mu)\sigma_1 + \cdots + q_1(\mu)\left(\frac{\sigma_2}{3}\right)^{1/3}\right. \\
&\left. + q_2(\mu)\left(\frac{\sigma_2}{3}\right)^{2/3} + q_3(\mu)\left(\frac{\sigma_2}{3}\right) + \cdots\right],
\end{aligned}
$$

branches off from P_4 and moves to a finite distance from P_4.

As the masses m_5 and m_6 increase from zero to positive values, the solution functions (14.5), (15.2) and (15.5) must be extended to include powers of $(\sigma_3/3)^{1/3}$ and $(\sigma_4/3)^{1/3}$. This can readily be accomplished by the substitution

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & y_2 & y_3 & y_4 & y_5 & y_6 & \mu & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ x_1 & x_2 & x_5 & x_6 & x_3 & x_4 & y_2 & y_5 & y_6 & y_3 & y_4 & \mu & \sigma_3 & \sigma_4 & \sigma_1 & \sigma_2 \end{pmatrix},$$

which leaves the system (13.1), for $n=6$, unchanged.

16. **The solution functions for any n.** It is evident that the process carried out in the last section may be repeated until the system contains any finite number of masses. Concerning the points of libration at any stage, we have the following theorem.

THEOREM 16.1. *As masses m_{2k-1} and m_{2k} are added to the system, by allowing them to increase from zero to some sufficiently small positive values at points P_{2k-1} and P_{2k} respectively, two points of libration P^*_{2k+1} and P^*_{2k+2} branch off from P_{2k-1} and P_{2k} respectively.*

In regard to the solution functions at any stage the following theorem is evident.

THEOREM 16.2. *As masses m_{2k+1} and m_{2k+2} increase from zero to some sufficiently small positive values, the solution functions x_i and y_i $(i=1, 2, \cdots, 2k+2)$ vary continuously and may be expressed as power series in $(\sigma_{2k-1}/3)^{1/3}$ and $(\sigma_{2k}/3)^{1/3}$, where σ_{2k-1} and σ_{2k} are the mass ratios m_{2k+1}/m_1 and m_{2k+2}/m_1 respectively.*

As pointed out in §14, all the solution functions for $n=4$ are power series in σ_1 and σ_2. This is due to the fact that equations (13.1) are invariant under the transformations

$$\begin{pmatrix} x_i & y_i & \sigma_{i-2} \\ x_j & \pm y_j & \sigma_{j-2} \end{pmatrix}, \qquad i, j = 3, 4, \cdots, n,$$

where the sign is to be taken "+" if i, j are both even or both odd, otherwise "−," and also to the fact that the solution functions if $n=4$ are power series in σ_1. For $n>4$ we note that only x_1, x_2 and y_2 are power series in σ_1, and hence only these three functions are power series in the mass ratios $\sigma_1, \sigma_2, \cdots, \sigma_{n-2}$. Since x_5 and y_5 are power series in $(\sigma_1/3)^{1/3}$, it follows that x_i and y_i $(i=3, 4, \cdots, n)$ are power series in $(\sigma_1/3)^{1/3}, (\sigma_2/3)^{1/3}, \cdots, (\sigma_{n-2}/3)^{1/3}$. It should be noted, however, that x_i and y_i $(i=3, 4, \cdots, n)$ may be power series in $(\sigma_1/3)^{1/3}, (\sigma_2/3)^{1/3}, \cdots, (\sigma_{n-3}/3)^{1/3}, (\sigma_{n-2}/3)$, but this solution does not have the property stated in Theorem 5.1, for as m_{n-2} vanishes P_{n-1} and P_n coincide.

Let $(\sigma_i/3)^{1/3}=\tau_i^{1/3}$. In case the solution functions are power series in $\tau_i^{1/3}$, $i=1, 2, \cdots, n-2$, the following functions have all the properties stated in theorems of §§5, 6, 7, 8 and constitute a solution of (13.1).

$$
\begin{aligned}
x_1 &= H_{1,0} + H_{1,1}\sum \tau_{2j-1}^{1/3} + \overline{H}_{1,1}\sum \tau_{2j}^{1/3} + H_{1,2}\sum \tau_{2j-1}^{2/3} + \overline{H}_{1,2}\sum \tau_{2j}^{2/3} \\
&\quad + H_{1,3}\sum \tau_{2j-1} + \overline{H}_{1,3}\sum \tau_{2j} + \cdots, \\
y_1 &= K_{1,0} + K_{1,1}\sum \tau_{2j-1}^{1/3} + \overline{K}_{1,1}\sum \tau_{2j}^{1/3} + K_{1,2}\sum \tau_{2j-1}^{2/3} + \overline{K}_{1,2}\sum \tau_{2j}^{2/3} \\
&\quad + K_{1,3}\sum \tau_{2j-1} + \overline{K}_{1,3}\sum \tau_{2j} + \cdots, \\
x_2 &= H_{2,0} + H_{2,1}\sum \tau_{2j-1}^{1/3} + \overline{H}_{2,1}\sum \tau_{2j}^{1/3} + H_{2,2}\sum \tau_{2j-1}^{2/3} + \overline{H}_{2,2}\sum \tau_{2j}^{2/3} \\
&\quad + H_{2,3}\sum \tau_{2j-1} + \overline{H}_{2,3}\sum \tau_{2j} + \cdots, \\
y_2 &= K_{2,0} + K_{2,1}\sum \tau_{2j-1}^{1/3} + \overline{K}_{2,1}\sum \tau_{2j}^{1/3} + K_{2,2}\sum \tau_{2j-1}^{2/3} + \overline{K}_{2,2}\sum \tau_{2j}^{2/3} \\
&\quad + K_{2,3}\sum \tau_{2j-1} + \overline{K}_{2,3}\sum \tau_{2j} + \cdots, \qquad (16.1)\\
x_{2i+1} &= H_{2i+1,0} + H_{2i+1,1}\sum{}' \tau_{2j-1}^{1/3} + \overline{H}_{2i+1,1}\sum{}' \tau_{2j}^{1/3} + H_{2i+1,2}\sum{}' \tau_{2j-1}^{2/3} \\
&\quad + \overline{H}_{2i+1,2}\sum{}' \tau_{2j}^{2/3} + H_{2i+1,3}\sum \tau_{2j-1} + \overline{H}_{2i+1,3}\sum \tau_{2j} + \cdots, \\
y_{2i+1} &= K_{2i+1,0} + K_{2i+1,1}\sum{}' \tau_{2j-1}^{1/3} + \overline{K}_{2i+1,1}\sum{}' \tau_{2j}^{1/3} + K_{2i+1,2}\sum{}' \tau_{2j-1}^{2/3} \\
&\quad + \overline{K}_{2i+1,2}\sum{}' \tau_{2j}^{2/3} + K_{2i+1,3}\sum \tau_{2j-1} + \overline{K}_{2i+1,3}\sum \tau_{2j} + \cdots, \\
x_{2i+2} &= H_{2i+2,0} + H_{2i+2,1}\sum{}' \tau_{2j-1}^{1/3} + \overline{H}_{2i+2,1}\sum{}' \tau_{2j}^{1/3} + H_{2i+2,2}\sum{}' \tau_{2j-1}^{2/3} \\
&\quad + \overline{H}_{2i+2,2}\sum{}' \tau_{2j}^{2/3} + H_{2i+2,3}\sum \tau_{2j-1} + \overline{H}_{2i+2,3}\sum \tau_{2j} + \cdots, \\
y_{2i+2} &= K_{2i+2,0} + K_{2i+2,1}\sum{}' \tau_{2j-1}^{1/3} + \overline{K}_{2i+2,1}\sum{}' \tau_{2j}^{1/3} + K_{2i+2,2}\sum{}' \tau_{2j-1}^{2/3} \\
&\quad + \overline{K}_{2i+2,2}\sum{}' \tau_{2j}^{2/3} + K_{2i+2,3}\sum \tau_{2j-1} + \overline{K}_{2i+2,3}\sum \tau_{2j} + \cdots,
\end{aligned}
$$

where $\sum$ denotes the sum over j from 1 to the largest integer in $(n-2)/2$, and $\sum'$ the same sum over j except for $j=i$. The H's and K's are functions of μ and the nonzero ones are

$$
\begin{aligned}
H_{1,0} &= \mu(1+\mu)^{-2/3}, \qquad H_{1,3} = \overline{H}_{1,3} = \frac{3-\mu}{2}(1+\mu)^{-5/3}, \\
H_{2,0} &= -(1+\mu)^{-2/3}, \\
H_{2,3} &= \overline{H}_{2,3} = \frac{1-3\mu}{2}(1+\mu)^{-5/3}, \qquad K_{2,3} = -\overline{K}_{2,3} = -\frac{3}{\mu}(1+\mu)^{1/3}, \\
H_{2i+1,0} &= H_{2i+2,0} = -\frac{1-\mu}{2}(1+\mu)^{-2/3}, \\
K_{2i+1,0} &= -K_{2i+2,0} = \frac{3^{1/2}}{2}(1+\mu)^{1/3},
\end{aligned}
$$

$$H_{2i+1,1} = \overline{H}_{2i+2,1} = \frac{1}{2} - \frac{1}{4}\mu + \cdots,$$

$$K_{2i+1,1} = -\overline{K}_{2i+2,1} = \frac{3^{1/2}}{2}\left(-1 - \frac{1}{2}\mu + \cdots\right),$$

$$H_{2i+1,2} = \overline{H}_{2i+2,2} = -\frac{1}{6} + \frac{125}{144}\mu + \cdots,$$

$$K_{2i+1,2} = -\overline{K}_{2i+2,2} = \frac{3^{1/2}}{2}\left(\frac{1}{3} + \frac{25}{72}\mu + \cdots\right),$$

$$H_{2i+1,3} = \overline{H}_{2i+2,3} = -\frac{9 + 14\mu + 13\mu^2}{4\mu}(1+\mu)^{-5/3},$$

$$K_{2i+2,3} = -\overline{K}_{2i+2,3} = -\frac{3^{1/2}\ (3+\mu)}{4\mu}(1+\mu)^{-2/3},$$

$$H_{2i+2,3} = \overline{H}_{2i+1,3} = \frac{3^{1/2}}{9\mu} + \frac{5}{4\mu} - \frac{2(3)^{1/2}}{27} + \frac{29}{12} - \frac{3^{1/2}}{27}\mu - \frac{7}{3}\mu + \cdots,$$

$$K_{2i+2,3} = -\overline{K}_{2i+1,3}$$
$$= -\frac{3^{1/2}}{2}\left(\frac{2(3)^{1/2}}{27\mu} + \frac{5}{6\mu} + \frac{8(3)^{1/2}}{81} + \frac{11}{18} + \frac{2(3)^{1/2}}{27}\mu - \frac{8}{9}\mu + \cdots\right).$$

Since (16.1) is a solution of (13.1), and hence of (2.1), for values of $\sigma_1, \sigma_2, \cdots, \sigma_{n-2}$ for which the series converge, and these values may be attained by choosing m_1 sufficiently large, we may state Theorem 12.2 as follows:

THEOREM 16.3. *Given $n-1$ arbitrary finite masses it is possible to choose an additional mass sufficiently large and an angular velocity sufficiently great, so that the n masses form a noncollinear plane permanent configuration.*

17. **The polygon formed by the P_i.** A diagram illustrates clearly the properties of a solution. In case $n=1$ the polygon, Fig. 3a, consists of one point P_1 at the origin. There is one point of libration P_2^* at a distance $(\lambda/m_1)^{1/3}=1$ from P_1. As a zero mass m_2 is placed at P_2^* and allowed to increase to a positive value, so that $(m_2/m_1)=\mu>0$, the configuration for $n=2$ is obtained. The polygon now consists of the line segment P_1P_2, Fig. 3b, with two points of libration P_3^* and P_4^* not collinear(7) with P_1 and P_2. These points are the vertices of equilateral triangles with side $P_1P_2=(1+\mu)^{1/3}$.

As a zero mass m_3 is placed at P_3^* and allowed to increase to a posi-

(7) The points of libration collinear with P_1P_2 are obtained by solving the first three equations (2.1) for x_3, where $n=3$ and $y_1=y_2=y_3=m_3=0$. For fixed order x_1, x_2, x_3 this is equivalent to Lagrange's quintic, cf. Lagrange [8, p. 277], or Tisserand [15, p. 155].

tive value, so that $(m_3/m_1)=\sigma_1>0$, the configuration for $n=3$ is obtained. The polygon in this case is the triangle $P_1P_2P_3$, Fig. 3c, which by the Theorem of Lagrange is an equilateral triangle of side $(1+\mu+\sigma_1)^{1/3}$. There are at least two(8) points of libration P_4^* and P_5^*. In general there is an $(n+2)$-sided polygon consisting of n points P_i and two libration points P_{n+1}^* and P_{n+2}^*.

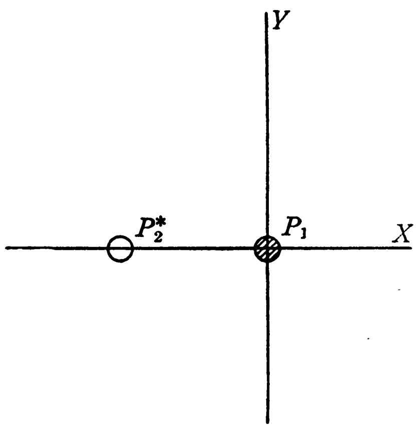

FIG. 3a

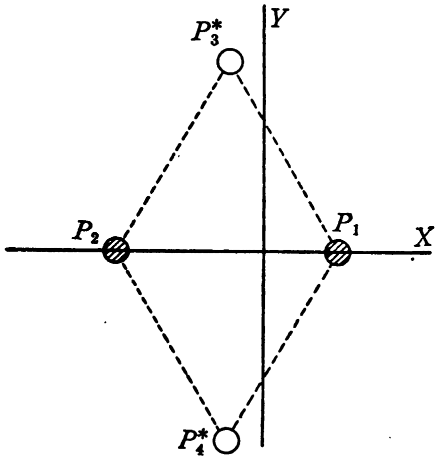

FIG. 3b

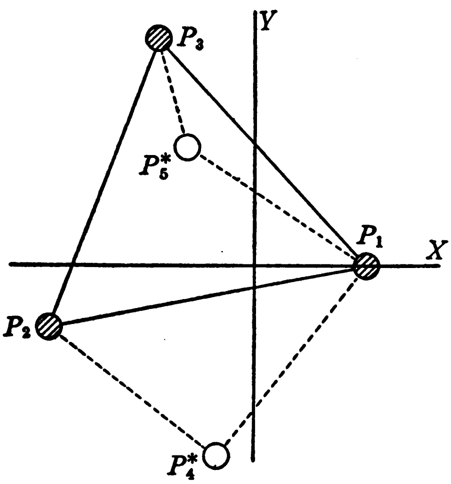

FIG. 3c

From the solution functions (16.1) we observe an interesting property which we state in the following theorem.

(8) According to Henrichsen [6], if the three masses are equal there are ten points of libration.

THEOREM 17.1. *As the masses m_i ($i=3, 4, \cdots, n$), located respectively at points P_i, approach zero the points P_i cluster about the two libration points for the two body problem.*

The geometric properties of these n-sided polygons, whether they are convex or concave, have not been determined. Furthermore, the possibility of removing the restrictions on the mass ratios seems to depend upon whether the determinant D_n in §12 is different from zero for all values of the masses.

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